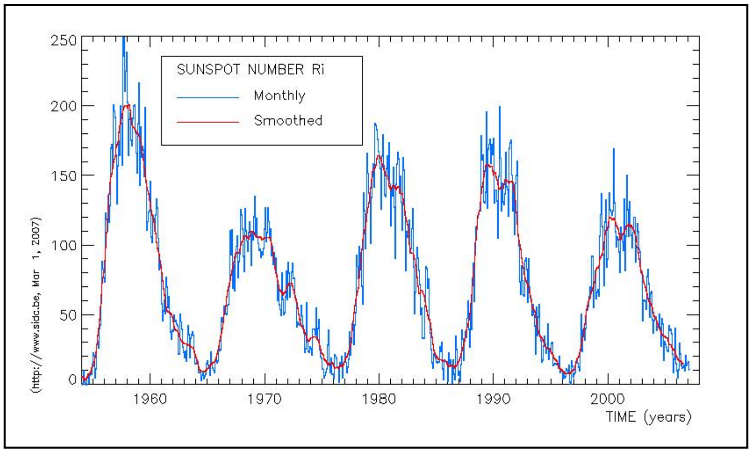
Matrix operations



Sunspots come and go in a roughly 11-year cycle. Astronomers measure the symmetry of these cycles by comparing the first 4 years with the last 4 years. If the cycles are exactly symmetric, the corresponding differences will be exactly zero.

Matrix A
Sunspot numbers at start of cycle.

	Year 1	Year 2	Year 3	Year 4
Cycle 23	**21**	**64**	**93**	**119**
Cycle 22	**13**	**29**	**100**	**157**
Cycle 21	**12**	**27**	**92**	**155**
Cycle 20	**15**	**47**	**93**	**106**

Matrix B
Sunspot numbers at end of cycle

	Year 11	Year 10	Year 9	Year 8
Cycle 23	**8**	**15**	**29**	**40**
Cycle 22	**8**	**17**	**30**	**54**
Cycle 21	**15**	**34**	**38**	**64**
Cycle 20	**10**	**28**	**38**	**54**

Problem 1 - Compute the average of the sunspot numbers for each cycle according to **C** = (**A** + **B**)/2.

Problem 2 - Compute the average difference of the sunspot numbers for the beginning and end of each cycle according to **D** = (**A** - **B**)/2.

Problem 3 – Are the cycles symmetric?

Problem 1 - Compute the average of the sunspot numbers for each cycle according to **C** = (**A** + **B**)/2. Answer:

$$C = \frac{1}{2}\begin{pmatrix} 21 & 64 & 93 & 119 \\ 13 & 29 & 100 & 157 \\ 12 & 27 & 92 & 155 \\ 15 & 47 & 93 & 106 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 & 15 & 29 & 40 \\ 8 & 17 & 30 & 54 \\ 15 & 34 & 38 & 64 \\ 10 & 28 & 38 & 54 \end{pmatrix}$$

$$C = \begin{pmatrix} 14.5 & 39.5 & 61.0 & 79.5 \\ 10.5 & 23.0 & 65.0 & 105.5 \\ 13.5 & 30.5 & 65.0 & 219.0 \\ 12.5 & 37.5 & 65.5 & 80.0 \end{pmatrix}$$

Problem 2 - Compute the average difference of the sunspot numbers for the beginning and end of each cycle according to **D** = (**A** - **B**)/2.

$$D = \frac{1}{2}\begin{pmatrix} 21 & 64 & 93 & 119 \\ 13 & 29 & 100 & 157 \\ 12 & 27 & 92 & 155 \\ 15 & 47 & 93 & 106 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 8 & 15 & 29 & 40 \\ 8 & 17 & 30 & 54 \\ 15 & 34 & 38 & 64 \\ 10 & 28 & 38 & 54 \end{pmatrix}$$

$$D = \begin{pmatrix} +6.5 & +24.5 & +32.0 & +39.5 \\ +2.5 & +6.0 & +35.0 & +66.5 \\ -1.5 & -3.5 & +27.0 & +45.5 \\ +2.5 & +9.5 & +27.5 & +26.0 \end{pmatrix}$$

Problem 3 – Are the cycles symmetric?

Answer: From the values in **D** we can conclude that the cycles are not symmetric, and from the large number of positive differences, that the start of each cycle has more spots than the corresponding end of each cycle.

Matrix operations

Depending on the type of star, its luminosity class, and its distance from Earth, stars appear at many different brightnesses in the sky.

Astronomers measure star brightness using an ancient magnitude scale designed by Hipparchus that ranks the star by its brightness so that a First Ranked star with a magnitude of +1.0 is 2.512 times brighter than a Second ranked star with a magnitude of +2.0.

Matrix M
Absolute magnitudes of each star and class

	MI	MII	MIII	MV
A0	-7.1	-3.1	-0.2	+0.7
F0	-8.2	-2.3	+1.2	+2.6
G0	-7.5	-2.1	+1.1	+4.4
K0	-7.5	-2.1	+0.5	+5.9

Matrix D
Distance modulus for each star and class

	DI	DII	DIII	DV
A0	+1.5	+1.5	+1.5	+1.5
F0	+1.5	+1.5	+1.5	+1.5
G0	+1.5	+1.5	+1.5	+1.5
K0	+1.5	+1.5	+1.5	+1.5

Problem 1 – An astronomer wants to determine the apparent magnitude, **A100**, for each star type (A0, F0, G0 and K0) and star class (I, II, III and V) at a distance of 100 light years. The formula is **A100** = **M** – 5 + 5**D.** What is the apparent magnitude matrix, **A100**, for these stars?

Problem 2 – The apparent magnitudes at a distance of 1,000 light years are given by **A1000** = **A100** + 2.4. A) How bright would the stars be at this distance? B) How bright would a sun-like star of type G0 and class V be at this distance?

Problem 1 – Answer: The formula is **A100** = **M** – 5 + 5**D.** What is the apparent magnitude matrix, **A100**, for these stars?

$$A100 = \begin{pmatrix} -7.1 & -3.1 & -0.2 & +0.7 \\ -8.2 & -2.3 & +1.2 & +2.6 \\ -7.5 & -2.1 & +1.1 & +4.4 \\ -7.5 & -2.1 & +0.5 & +5.9 \end{pmatrix} - 5 + 5\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix}$$

Note: The first cell becomes -7.1 - 5 + 5(1.5) = -4.6 and subsequent cells evaluated in a similar manner.

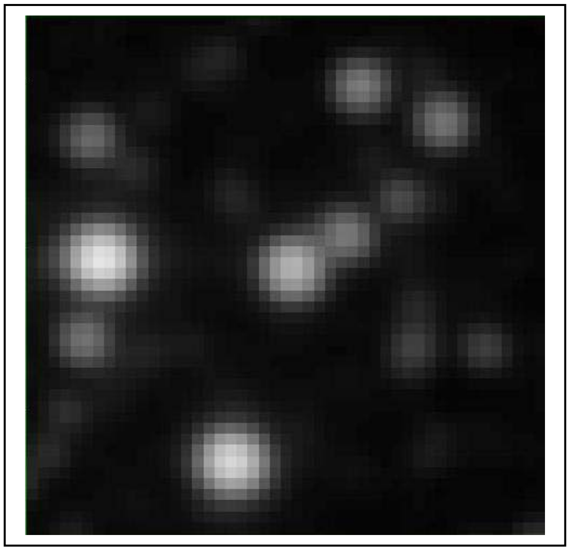
$$A100 = \begin{pmatrix} -4.6 & -0.6 & +2.3 & +3.2 \\ -5.7 & +0.2 & +3.7 & +5.1 \\ -5.0 & +0.4 & +3.6 & +6.9 \\ -5.0 & +0.4 & +3.0 & +8.4 \end{pmatrix}$$

Problem 2 – A) How bright would the stars be at this distance? B) How bright would a sun-like star of type G0 and class V be at this distance?
Answer:

$$A1000 = \begin{pmatrix} -2.2 & +1.8 & +4.7 & +5.6 \\ -3.3 & +2.6 & +6.1 & +7.5 \\ -2.6 & +2.8 & +6.0 & +9.3 \\ -2.6 & +2.8 & +5.4 & +10.8 \end{pmatrix}$$

B) From the table, G0 is the third row and class V is the last column so the brightness of this star would be **+9.3**

Matrix operations



Astronomical photography is based upon the design of high-tech cameras that use millions of individual sensors. The sensors measure the brightness of specific directions of the sky. This gives these images a pixelated appearance.

Astronomers manipulate digital images as large matrices of data. They operate on these image matrices to calibrate, correct and enhance the clarity and accuracy of the digital data. This also leads to some spectacular photographs too!

	Col. 1	Col. 2	Col. 3	Col. 4
Row 1	**64**	**64**	**64**	**65**
Row 2	**65**	**66**	**66**	**84**
Row 3	**67**	**215**	**67**	**67**
Row 4	**67**	**68**	**67**	**67**

The above matrix of numbers represents the intensity values that were measured in a region of the sky that spanned 4x4 = 16 pixels in area. Each number indicates the digital value that corresponds to the instrument's voltage measurement in specific pixels. The astronomer wants to subtract from the image, **I**, the values in each pixel that correspond to the light from the sky, **S**, to isolate the light from the two stars in the field. He also wants to convert the numbers from 'instrument numbers' to actual brightness values of the physical object in the sky by using the calibration constant '4.5'. The end result will be a 'cleaned' image, **C**, that is accurately calibrated so that actual astronomical research can be conducted.

Problem 1 – The contribution from the sky has been modeled by the matrix S given by

$$S = \begin{pmatrix} 63 & 63 & 63 & 63 \\ 64 & 64 & 64 & 64 \\ 65 & 65 & 65 & 65 \\ 66 & 66 & 66 & 66 \end{pmatrix}$$

Create a calibrated image by performing the operation **C** = 4.5 x (**I** - **S**).

Problem 2 - Where are the two bright stars located in the image?

Problem 1 - Answer:

$$\begin{pmatrix} 64 & 64 & 64 & 65 \\ 65 & 66 & 66 & 84 \\ 67 & 215 & 67 & 67 \\ 67 & 68 & 67 & 67 \end{pmatrix} - \begin{pmatrix} 63 & 63 & 63 & 63 \\ 64 & 64 & 64 & 64 \\ 65 & 65 & 65 & 65 \\ 66 & 66 & 66 & 66 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 20 \\ 2 & 150 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

Then $C = 4.5 \times \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 20 \\ 2 & 150 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{pmatrix}$

$C = \begin{pmatrix} 4.5 & 4.5 & 4.5 & 9 \\ 4.5 & 9 & 9 & 90 \\ 9 & 675 & 9 & 9 \\ 4.5 & 9 & 4.5 & 4.5 \end{pmatrix}$ is the calibrated image

Problem 2 - Where are the two bright stars located in the image?
Answer: **The two bright stars are located in Column 2 row 3 ('675') and column 4 row 2 ('90')**

	U-B	B-V	V
Altair	+0.08	+0.22	+0.76
Sun	+0.13	+0.65	-26.7
Antares	-0.84	+1.81	+1.0
Aludra	-0.73	-0.07	+2.42
Proxima Centauri	+1.49	+1.97	+11.05

Astronomers measure the brightness of stars as viewed through different filters. In the visible spectrum, these filters are called the U, B and V bands. By performing simple operations on these brightnesses, measured in terms of stellar magnitudes, the temperature and other properties of the stars can be determined.

Problem 1 - From the table above, create a new table that gives the following information (For example, for the star Antares, V = +1.00 and B-V = +1.81 so B = +2.81)

	U	B	V
Altair			+0.76
Sun			-26.7
Antares		**+2.81**	+1.0
Aludra			+2.42
Proxima Centauri			+11.05

Problem 2 - An astronomer wants to determine the brightness of each star in the three filters U, B and V by recalculating their brightness at a common distance of 32.6 light years (10 parsecs). To do this, he takes the matrix defined by the numbers in the table in Problem 1, called **D**, and performs the following operation: **N** = **D** + **S** where **S** is the 'shift' matrix defined by:

$$S = \begin{pmatrix} +1.5 & +1.5 & +1.5 \\ +31.6 & +31.6 & +31.6 \\ -5.6 & -5.6 & -5.6 \\ -9.4 & -9.4 & -9.4 \\ +4.4 & +4.4 & +4.4 \end{pmatrix}$$

What is the new matrix of star brightnesses **N**, and the corresponding new table?

Problem 1 - Answer:

	U	B	V
Altair	**+1.06**	**+0.98**	+0.76
Sun	**-25.92**	**-26.05**	-26.7
Antares	**+1.97**	**+2.81**	+1.0
Aludra	**+1.62**	**+2.35**	+2.42
Proxima Centauri	**+14.51**	**+13.02**	+11.05

Problem 2 – Answer:

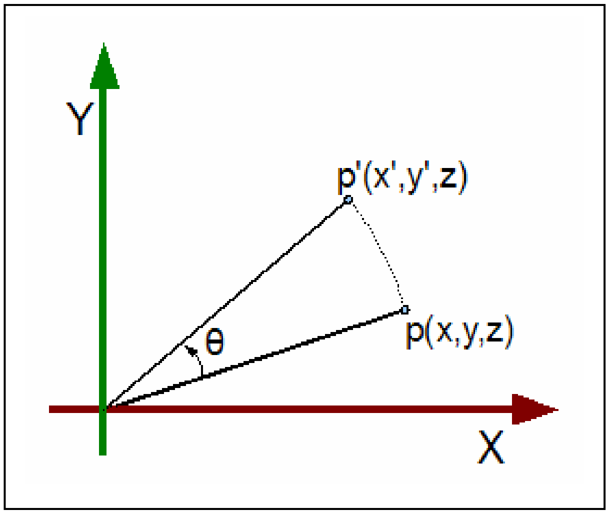
$$N = \begin{pmatrix} +1.06 & +0.98 & +0.76 \\ -25.92 & -26.05 & -26.7 \\ +1.97 & +2.81 & +1.0 \\ +1.62 & +2.35 & +2.42 \\ +14.51 & +13.02 & +11.05 \end{pmatrix} + \begin{pmatrix} +1.5 & +1.5 & +1.5 \\ +31.6 & +31.6 & +31.6 \\ -5.6 & -5.6 & -5.6 \\ -9.4 & -9.4 & -9.4 \\ +4.4 & +4.4 & +4.4 \end{pmatrix} = \begin{pmatrix} +2.6 & +2.5 & +2.3 \\ +5.7 & +5.5 & +4.9 \\ -3.6 & -2.8 & -4.6 \\ -7.8 & -7.0 & -7.0 \\ +18.9 & +17.5 & +15.5 \end{pmatrix}$$

So:

Table of Star Brightnesses at 32.6 light years

	U	B	V
Altair	+2.6	+2.5	+2.3
Sun	+5.7	+5.5	+4.9
Antares	-3.6	-2.8	-4.6
Aludra	-7.8	-7.0	-7.0
Proxima Centauri	+18.9	+17.5	+15.5

Multiplying Matrices



Matrix multiplication is used when rotating the coordinates of a point in one coordinate system p(x,y,z) into another coordinate system p(x', y', z'). The general formula is

p' = **R** p

If **R** represents the matrix for a 10-degree clockwise rotation, then **R**x**R** represents the rotation matrix for a 20-degree clockwise rotation, and **R**x**R**x**R** represents a 30-degree rotation, and so on.

Problem 1 – If the rotation matrix for a 90-degree clockwise rotation is given by **R**, write the four matrix equations for the coordinates of a point P(x,y) after each rotation of 0, 90, 180, 270, 360 is applied, where the final coordinates are indicated by P'.

Problem 2 - The rotation matrix, R, for a 90-degree counter-clockwise rotation is given by

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What are the coordinates of the point P(+25, +15) after each of the rotations are applied?

Problem 1 – If the rotation matrix for a 90-degree clockwise rotation is given by **R**, write the four matrix equations for the coordinates of a point P(x,y) after each rotation of 0, 90, 180, 270, 360 is applied, where the final coordinates are indicated by P'.

Answer:

0 degrees: P' = P

90 degrees: P' = **R** P

180 degrees: P' = **R** x **R** P

270 degrees: P' = **R** x **R** x **R** P

360 degrees: P' = **R** x **R** x **R** x **R** P

Problem 2 - The rotation matrix, R, for a 90-degree counter-clockwise rotation is given by

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What are the coordinates of the point P(+25, +15) after each of the rotations are applied?

0 degrees: P' = P so **P' = (+25, +15)**

90 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; **P' = (-15, +25)**

180 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; P' = $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$P ; **P' = (-25, -15)**

270 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ P ; P' = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$P ; **P' = (+15, -25)**

360 degrees: P' = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$P ;

P' = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$P ; **P' = (+25, +15)**

Multiplying Matrices

Although the mass of a body, in kilograms, does not vary, the quantity that we call 'weight' depends on the force of gravity acting on the given mass.

A simple relationship using matrices allows us to 'weigh' different bodies at differing distances from the surface of Earth.

A =

	H=0	H=10 km	H=500km
Mercury	363	360	249
Moon	162	160	97
Earth	982	979	844
Mars	374	372	284

Acceleration units are centimeters/sec^2

M=

	Mass
Golfball	0.05
Human	70.0
Space Station	246,000

Mass units are kilograms

Problem 1 – The weight of a body, in pounds, is given by **W** = 0.0022 **A M**, where A is the acceleration matrix for gravity for each of the bodies, at three different altitudes above the surface, and m is the mass, in kilograms, of the three test bodies being studies. What are the weights of each object at the corresponding altitudes?

Problem 1 – The weight of a body, in pounds, is given by **W** = 0.0022 **A M**, where A is the acceleration matrix for gravity for each of the bodies, at three different altitudes above the surface, and **M** is the mass, in kilograms, of the three test bodies being studies. What are the weights of each object at the corresponding altitudes?

$$A = \begin{pmatrix} 363 & 360 & 249 \\ 162 & 160 & 97 \\ 982 & 979 & 844 \\ 374 & 372 & 282 \end{pmatrix} \qquad M = \begin{pmatrix} 0.05 \\ 70.0 \\ 246,000 \end{pmatrix}$$

Acceleration units are centimeters/sec^2 Mass in kilograms

W = 0.0022 A M

$$W = 0.0022 \begin{pmatrix} 363 & 360 & 249 \\ 162 & 160 & 97 \\ 982 & 979 & 844 \\ 374 & 372 & 282 \end{pmatrix} \begin{pmatrix} 0.05 \\ 70.0 \\ 246,000 \end{pmatrix}$$

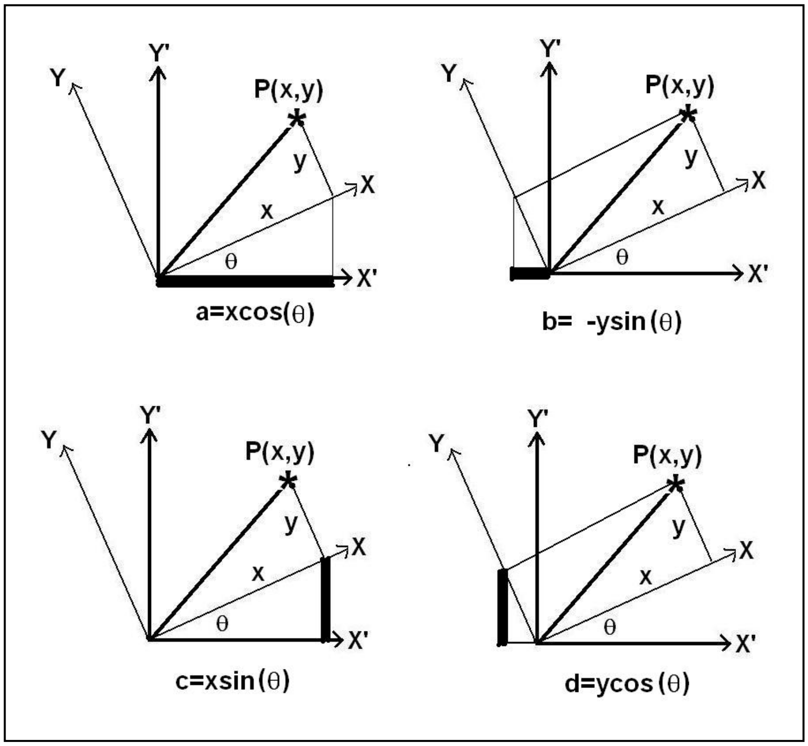
Example for first cell: 0.0022 (363)0.05 = 0.04

So including labels:
W =

	H=0	H=10 km	H=500km
Mercury	0.04	55.4	134,759
Moon	0.02	24.5	52,496
Earth	0.1	150.8	456,773
Mars	0.04	57.3	153,701

Weight units in pounds.

Note: At an altitude of 500 km, the object would actually be in orbit and so A = 0, which means that W = 0 or 'weightless'. This is where the acceleration of gravity towards the center of earth is exactly equal to the local centripetal acceleration of the orbiting spacecraft. These oppositely-directed forces yield a net-zero acceleration so it would be weightless.



Although we could create a list of all possible rotation matrices for each possible angle, it is far more economical to use trigonometric relationships to make the process more general.

The four sketches to the left illustrate the origin of the various factors a, b, c and d, (highlighted) that define the general coordinate transformation in Cartesian coordinates between (x,y) and (x',y') where (x,y) has been rotated by an angle, θ, with respect to (x',y'):

$$X' = aX + bY$$
$$Y' = cX + dY$$

It is always a bit confusing, at first, to see why the 'a term' has a sign opposite to the others, but look at the top-right figure. The positive-y axis leans over the negative-x axis, so any positive value for y will be mapped into a negative number for its horizontal x-projection. That's why when you sum-up the parts that make up the total x' value, you get one part from the positive-x projection, xcos(θ), and then you have to flip the sign before you add the part from the positive-y projection, -ysin(θ). If you just left it as +ysin(θ), that would be geometrically wrong, because the positive-y axis is definitely NOT leaning to the right into the First Quadrant.

Problem 1 - Write the complete rotation of (x,y) into (x',y') as two linear equations.

Problem 2 - Write the rotation as a matrix equation X' = **R(θ)** X

Problem 3 - What is the rotation matrix for a rotation of A) +90 degrees clockwise? B) + 90 degrees counter-clockwise? C) 180-degrees clockwise?

Problem 4 - What is the exact rotation matrix for a rotation of 60 degrees clockwise?

Problem 5 - What is the inverse matrix $\mathbf{R(\theta)}^{-1}$?

Problem 6 - Show that, for all angles α and β: **R(α) R(β)** is not the same as **R(β) R(α)**.

Problem 1 - Answer:

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

Problem 2 - Answer:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Problem 3 - Answer

A) $R(+90) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ B) $R(-90) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ C) $R(+180) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Problem 4 - Answer: For exact answers, do not evaluate fractions or square-roots:

$$R(+60) = \begin{pmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Problem 5 - Answer:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad R(\theta)^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

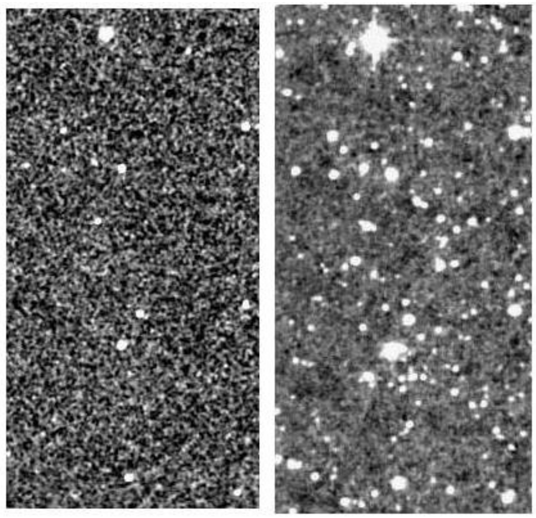
So $R(\theta)^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

Problem 6 - Show that, for all angles α and β: **R(α) R(β)** is not the same as **R(β) R(α)**.

$$R(\alpha)R(\beta) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix}$$

$$R(\beta)R(\alpha) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix}$$

Although the diagonal terms are symmetric in α and β, the off-diagonal terms are not. This demonstrates that rotation matrices do not commute under multiplication so that in general AB does not equal BA. The order of operation is important in matrix mathematics.



Images taken by the 2MASS sky survey.
(Left) Raw image exposed for 8 seconds.
(Right) 2,050 raw images stacked and averaged together to detect fainter stars.

Astronomers often take dozens, or even thousands of digital images of the same region of the sky in order to 'add them up' and detect very faint objects. This 'stacking' of images requires that the same pixels be added together to form the average. This can be a problem if the telescope, or satellite, is in motion.

One kind of motion is called pure rotation. Every image is tagged by its orientation angle so that, when the image is later processed, it can be properly averaged into the other images in the stack.

Suppose that the address of a pixel in the stacked image is given by (X',Y') and the address of the corresponding pixel in the raw image is (X,Y) observed at an angle, θ, with respect to the stacked image coordinates. The relationship between the two coordinate systems is just:

$$X' = X\cos(\theta) - Y\sin(\theta)$$
$$Y' = X\sin(\theta) + Y\cos(\theta)$$

Problem 1 - An astronomer wants to combine the data from pixel P(x,y)=(245,3690) in a raw image, with the averaged data in the stacked image. What will be the 'destination' address of the data pixel in the stacked image, P(x',y') if the data in the raw image is rotated 5 degrees clockwise relative to the stacked image? (Note: It is helpful to draw a picture to keep track of P and P')

Problem 2 - An astronomer wants to add an additional raw image to an image stack of 25 images, where the raw image pixels have the following intensities:

P(497,1030)=90.0	P(497,1031) = 85.0
P(498,1030)=35.0	P(498,1031) = 20.0

A) For what rotation angle do these four raw pixels coincide with the stacked pixels whose intensities are:

S(358,1086) = 93.5	S(358,1087) = 87.2
S(359,1086) = 32.4	S(359,1087) = 21.2

B) What will be the new averages for the stacked image pixels?

Problem 1 - **Answer:**

X' = (245) cos(5) - (3690)sin(5) = 244.1 - 321.6 = -77.5 = -77
Y' = (245)sin(5) + (3690)cos(5) = 21.4 + 3676.0 = 3697.4 = 3697.

So the data from P(245,3690) in the raw image should be placed in pixel P'(-77,3697) in the stacked image.

Problem 2 - Answer: A) Students should realize that this represents solving a system of 2 equations in two unknowns 'sin(θ)' and 'cos(θ)'. Set up the equations as follows using any of the corresponding coordinate pairs P and S:

$$358 = 497\cos(\theta) - 1030\sin(\theta)$$
$$1086 = 1030\cos(\theta) + 497\sin(\theta)$$

As a matrix equation:

$$\begin{pmatrix} 358 \\ 1086 \end{pmatrix} = \begin{pmatrix} 497 & -1030 \\ 1030 & 497 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

The inverse matrix is $\frac{1}{(497)^2 + (1030)^2} \begin{pmatrix} 497 & 1030 \\ -1030 & 497 \end{pmatrix} = \begin{pmatrix} 0.000379 & 0.000788 \\ -0.000788 & 0.000379 \end{pmatrix}$

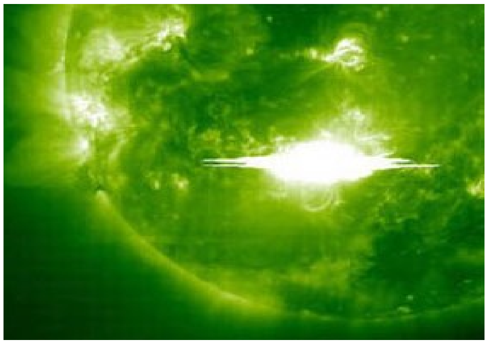
Then $\begin{pmatrix} 0.000379 & 0.000788 \\ -0.000788 & 0.000379 \end{pmatrix} \begin{pmatrix} 358 \\ 1086 \end{pmatrix} = \begin{pmatrix} \sin(\theta) \\ \cos(\theta) \end{pmatrix}$

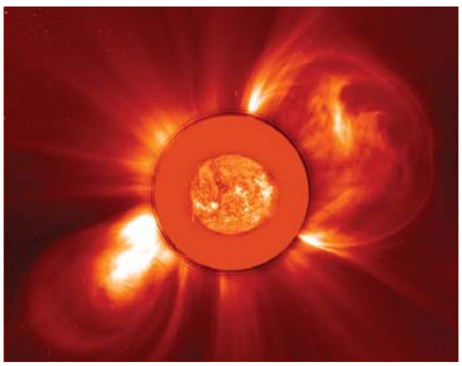
So cos(θ) = 0.1357+0.856 = 0.9917 or sin(θ)= -0.2821 + 0.4116= 0.1295
So either way, **θ = 7.4 degrees**. Answers near 7.5 degrees are acceptable.

B) You have to do a weighted average:

S(358,1086) = (93.5*25 + 90.0)/26 = **93.4**
S(358,1087) = (87.2*25 + 85.5)/26 = **87.1**
S(359,1086) = (32.4*25 + 35.0)/26= **32.5**
S(359,1087) = (21.2*25 + 20.0)/26 = **21.1**

Solving Systems Using Inverse Matrices





Solving a system of three equations in three unknowns can commonly be found in several space science and astronomy applications.

Solar flares are a frequent phenomenon on the sun, especially during the peaks of solar activity cycles. Over 21,000 can occur during an average solar cycle period of 11 years! In our first problem, you will determine the average intensity of three classes of flares ('C', 'M' and 'X') by using statistical information extracted from three solar activity (sunspot!) cycles.

During February 4 - 6, 2000 the peak month of Cycle 23 solar scientists tallied 37 C-class, 1 M-class and 1 X-class flares, for a total x-ray intensity of 705 mFU (1 mFU = 10^{-6} watts/m^2) .

During March 4 - 6, 1991 scientists tallied 15 C-class, 14 M-class and 4 X-class flares for a total x-ray intensity of 2775 mFU

During April 1 - 3, 2001 scientists tallied 5 C-class, 9 M-class and 4 X-class flares for a total x-ray intensity of 2475 mFU.

Problem 1: Use the above data to create a system of equations, solve them, and determine the average intensity of flares, to the nearest tenth, in each category (C, M and X) in units of mFU.

Answer Key:

After setting up the problems as a matrix, you might want to use the spiffy online matrix calculator at
http://www.bluebit.gr/matrix-calculator/

Problem 1:

The system of equations is

$$31\,C + 1\,M + 1\,X = 705$$
$$15\,C + 14\,M + 4\,X = 2775$$
$$5\,C + 9\,M + 4\,X = 2475$$

Matrix:

$$\begin{matrix} 31 & 1 & 1 \\ 15 & 14 & 4 \\ 5 & 9 & 4 \end{matrix}$$

Inverse:

$$\begin{matrix} 0.031 & 0.008 & -0.016 \\ -0.062 & 0.184 & -0.169 \\ 0.101 & -0.425 & 0.650 \end{matrix}$$

Solution:

C : 0.031 x 705 + 0.008 x 2775 -0.016 x 2475 = **4.5 mFU**
M: -0.062 x 705 + 0.185 x 2775 -0.169 x 2475 = **51.4 mFU**
X: 0.101 x 705 -0.425 x 2775 +0.650 x 2475 = **500.2 mFU**

Solving Systems Using Inverse Matrices

Solving a system of three equations in three unknowns can commonly be found in several space science and astronomy applications.

Communications satellites use electrical devices called transponders to relay TV and data transmissions from stations to satellite subscribers around the world.

There are two basic types: K-band transponders operate at frequencies of 11-15 GHz and C-band transponders operate at 3-7 GHz.

Satellites come in a variety of standard models, each having its own power requirements to operate its pointing and positioning systems. The following satellites use the same satellite model:

Satellite 1 : Anik F1
Total power = 15,000 watts
Number of K-band transponders = 48
Number of C-band transponders = 36

Satellite 2 : Galaxy IIIc
Total power = 14,900 watts
Number of K-band transponders = 53
Number of C-band transponders = 24

Satellite 3 : NSS-8
Total power = 16,760 watts
Number of K-band transponders = 56
Number of C-band transponders = 36

Problem 1: Use the data to determine the average power, to the nearest integer, of a K-band and a C-band transponder, and the satellite operating power, F, in watts.

Answer Key:

After setting up the problems as a matrix, you might want to use the spiffy online matrix calculator at

http://www.bluebit.gr/matrix-calculator/

Problem 1. Solving for satellite transponder power, K and C, and satellite operating power, F using 3 equations in three variables. From the satellite data

$$48\,K + 36\,C + F = 15{,}000$$
$$53\,K + 24\,C + F = 14{,}900$$
$$56\,K + 36\,C + F = 16{,}760$$

Matrix:
$$\begin{matrix} 48 & 36 & 1 \\ 53 & 24 & 1 \\ 56 & 36 & 1 \end{matrix}$$

Inverse:
$$\begin{matrix} -0.125 & 0.0 & 0.125 \\ 0.031 & -0.083 & 0.052 \\ 5.875 & 3.00 & -7.875 \end{matrix}$$

Solution = -0.125 x 15000 + 0.125 x 16,760 = K = **220 watts per K-band transponder**

0.031 x 15000 - 0.083 x 14,900 + 0.052 x 16,760 = C = **100 watts per C-band transponder**

5.875 x 15000 + 3 x 14,900 - 7.875 x 16,760 = F = **840 watts for the satellite operating power**

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad C^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Rotation matrices are a basic mathematical ingredient to photo imaging software (PaintShop, Adobe Illustrator, etc). A typical software menu lets you select by what angle you want to rotate an image. Because satellites spin, and spacecraft have to 'roll' or 'pitch' or 'yaw' to change their orientation in space, rotation matrices are a vital ingredient to space science.

In this problem, we are going to explore the properties of rotation matrices in 2-dimensions. Think of this as studying what happens to images in the x-y plane as they are rotated clockwise or counter-clockwise about the z-axis.

The original image has pixels arranged in a rectangular grid along the x and y axis denoted by the coordinate pairs (x,y). The final, rotated, image has a new set of pixel coordinates denoted by (x', y'). The matrix equation that relates the old and new coordinates is just

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{or} \qquad X' = \mathbf{R}X$$

For the simple case of 90-degree rotations, the rotation matrices, **R**, are shown to the left, along with their inverses.

Problem 1 - In terms of the initial (x,y) and final (x',y') coordinates, describe what each of the rotation matrices **I**, **A**, **B** and **C** does.

Problem 2 - For each matrix, **I**, **A**, **B** and **C**, what is the physical interpretation of the corresponding inverse matrix?

Problem 3 - Show that the matrix equation **AB** correctly represents a rotation of **A** followed by a rotation of **B**, but that the equation **A** + **B** does not.

Problem 4 - Compute the final result of $\mathbf{AA^{-1}}$ and explain what happens physically. What is a general rule relating a rotation matrix and its inverse?

Problem 5 - A spacecraft undergoes a complex series of rotations while moving to its next target to observe. The sequence of rotations is represented by $ABA^{-1}CB^{-1}$. How is the final coordinate system (x',y') related to the initial one (x,y) after the moves are completed?

Problem 1 - Answer: X' = I X yields (x',y') = (x,y) Rotates (x,y) by zero degrees
X' = AX yields (x',y') = (-y,x) Rotates (x,y) by 90 degrees clockwise
X' = BX yields (x',y') = (y,-x) Rotates (x,y) by 90 degrees counter-clockwise
X' = CX yields (x',y') = (-x,-y) Rotates (x,y) by 180 degrees clockwise

Problem 2 - Answer: X' = I^{-1} X yields (x',y') = (x,y) Rotates (x,y) by zero degrees
X' = A^{-1}X yields (x',y') = (y,-x) Rotates (x,y) by 90 degrees counter-clockwise
X' = B^{-1}X yields (x',y') = (-y,x) Rotates (x,y) by 90 degrees clockwise
X' = C^{-1}X yields (x',y') = (-x,-y) Rotates (x,y) by 180 degrees counter-clockwise

Problem 3 - Answer:

$AA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ implies two 90-degree clockwise rotations so (x',y') = (-x,-y) This is equivalent to 1, 180-degree clockwise rotation and so **AA** = **C**

$A + A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ which is the same as 2A so that (x',-y') = (2x,-2y) = 2(x,-y) and is not a rotation.

Problem 4 - Answer: $AA^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ A rotation of 90 clockwise followed by a rotation of 90 degrees **counter**-clockwise leaves the coordinates unchanged. The inverse rotation matrices represent the corresponding rotation matrix with the sign of the angle reversed.

Problem 5 - Write out the matrix products and evaluate from left to right:

$$\begin{aligned}
\mathrm{ABA^{-1}CB^{-1}CC^{-1}B} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

So (x',y') = (-x,-y) and this is just a rotation by 180 degrees from the original (x,y).

$\_\mathbf{w}\_MgO + \_\mathbf{x}\_Fe \rightarrow \_\mathbf{y}\_Fe_2O_3 + \_\mathbf{z}\_Mg$

Mg: 1 w + 0 x + 0 y = 1 z
Fe: 0 w + 1 x - 2 y = 0 z
O: 1 w + 0 x - 3 y = 0 z

$$A = \begin{pmatrix} 1, 0, & 0 \\ 0, 1, & -2 \\ 1, 0, & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \det(A)=3$$

$$\begin{pmatrix} w \\ x \\ y \end{pmatrix} = A^{-1}B\ (\det(A))$$

so w=3, x=2, y=1 and z=det(A)=3

$\mathbf{3}MgO + \mathbf{2}Fe \rightarrow \mathbf{1}Fe_2O_3 + \mathbf{3}Mg$

Matrix mathematics can be used to balance chemical reaction equations. Although this can be a tedious, but often entertaining, process for humans, it can be automated and 'solved' by using a computer program and matrix math. The example to the left shows the steps.

First re-write the equation with only one compound on the right-hand side.

Next, separate the chemical equation into one equation for each element.

Then create the two arrays, A and B, and compute the determinant of A.

Finally, solve the matrix equation for w, x, y and z taking the Absolute Values of all numbers and rounding them to the nearest integer.

Problem 1 - What integers will 'balance' the chemical reaction describing the combustion of gasoline in a car engine as follows:

___ C_8H_{18} + ___ O_2 $\rightarrow$ ___ CO_2 + ___ H_2O